Calculus 2014/15 Exercise Sheet 1

You should attempt all of these questions. When you run into difficulties, you can post on the Moodle discussion forum, go to a drop-in session, ask your tutor, email me … but don't give up! Please hand in your solutions to the starred questions (1a, 5, 7) to the blue mailbox in James Nucleus labelled Calculus/Algebra with your tutor's name on it, before the deadline shown on Moodle. Marked work will be returned in your tutorial.

Notation and terminology

Recall from your earlier studies: a function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. For example, if $f(x) = x^n$ for some $n \in \mathbb{N}$ then f is an odd function if n is an odd number, and f is an even function if f is an even number.

Notation: given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we construct their sum, difference and product in the obvious ways:

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x) \qquad (fg)(x) = f(x)g(x)$$

1. Suppose $f, g : \mathbb{R} \to \mathbb{R}$. Show that:
   - ⋆ (a) If f and g are both odd then $f+g$ and $f-g$ are odd and fg is even.
   - (b) If f and g are both even then $f+g$, $f-g$ and fg are all even.
   - (c) If f is even and g is odd then fg is odd.
   - (d) If f is both even and odd then f is identically zero (i.e. $f(x) = 0$ for all $x \in \mathbb{R}$).

2. Suppose $f : \mathbb{R} \to \mathbb{R}$. Follow these steps to write f as the sum of an odd function and an even function:
   - (a) Assume that there is such a decomposition, say $f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x)$ for all $x \in \mathbb{R}$.
   - (b) Write down the equation $f(-x) = f_{\text{odd}}(-x) + f_{\text{even}}(-x)$ and simplify it using the hypotheses that f_{odd} is odd and f_{even} is even.
   - (c) Solve two simultaneous equations to find formulae for $f_{\text{odd}}(x)$ and $f_{\text{even}}(x)$, in terms of $f(x)$ and $f(-x)$.
   - (d) Check that your formulae for f_{odd} and f_{even} really do describe an odd and an even function and that $f_{\text{odd}}(x) + f_{\text{even}}(x) = f(x)$ for all x.

   Why is this last step, checking that your answers work, necessary? Is it possible to find a different pair of odd and even functions whose sum is also f?

3. Suppose P is a polynomial, say

$$P(x) = \sum_{n=0}^{N} a_n x^n$$

We can write the odd-even decomposition of P as follows:

$$P_{\mathrm{odd}}(x) = \sum_{\substack{n=0 \\ n \text{ odd}}}^{N} a_n x^n \qquad P_{\mathrm{even}}(x) = \sum_{\substack{n=0 \\ n \text{ even}}}^{N} a_n x^n \qquad P(x) = P_{\mathrm{odd}}(x) + P_{\mathrm{even}}(x) \qquad (\text{all } x \in \mathbb{R})$$

Deduce from this that:

(a) P is even if and only if all the odd powers of x in $P(x)$ have coefficient zero.

(b) P is odd if and only if all the even powers of x in $P(x)$ have coefficient zero.

4. For $x \in \mathbb{R}$, $x \neq 1$, let
$$f(x) = \frac{x^2 - 1}{x - 1}.$$
Evaluate $\lim_{x \to 1} f(x)$. What can you say about $\lim_{y \to 1} f(y)$ and $\lim_{z \to 1} f(z)$?

★ 5. Each one of these formulae is intended to define a function from the largest possible subset of $\mathbb{R}$ to $\mathbb{R}$. What is its maximal domain, i.e. the largest subset of $\mathbb{R}$ on which the formula makes sense?
$$f_1(x) = x^2 + x - 1 \qquad f_2(y) = \frac{y^3 - y}{2y^2 + 1} \qquad f_3(z) = \frac{z^2 - 5z + 6}{z - 2}$$
$$f_4(u) = \frac{2u^2 + 5u - 11}{u^2 + 1} \qquad f_5(v) = \frac{2v + 3}{v^2 - 1}$$

6. Repeat the previous question with these formulae:
$$g_1(x) = 3x^3 - x + 4 \qquad g_2(y) = \frac{y^2 + y - 2}{y + 2} \qquad g_3(z) = \frac{2z^3 - 7z + c}{z^3 + 8}$$
Here, c is some unspecified real number.

★ 7. Using $f_1, f_2, \ldots, f_5$ as defined above, evaluate the following limits:
$$\text{(a) } \lim_{x \to 1} f_1(x) \qquad \text{(b) } \lim_{y \to 2} f_2(y) \qquad \text{(c) } \lim_{z \to 2} f_3(z)$$
$$\text{(d) } \lim_{u \to \infty} f_4(u) \qquad \text{(e) } \lim_{u \to -\infty} f_5(v)$$
In the cases where the maximal domain (which you found in the earlier question) is not the whole of $\mathbb{R}$, why does this not affect the existence or value of the limit?

8. Using g_1, g_2, g_3 as defined above, evaluate the following limits:
$$\text{(a) } \lim_{x \to 3} g_1(x) \qquad \text{(b) } \lim_{y \to 2} g_2(y) \qquad \text{(c) } \lim_{y \to -2} g_2(y)$$
$$\text{(d) } \lim_{z \to \infty} g_3(z) \qquad \text{(e) } \lim_{z \to -\infty} g_3(z)$$
In the cases where the maximal domain (which you found in the earlier question) is not the whole of $\mathbb{R}$, why does this not affect the existence or value of the limit?